

Combinatorial Interpretation of General Eulerian Numbers

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Abstract

Since 1950s, mathematicians have successfully interpreted the traditional Eulerian numbers and q -Eulerian numbers combinatorially. In this paper, the authors give a combinatorial interpretation to the general Eulerian numbers defined on general arithmetic progressions $\{a, a + d, a + 2d, \dots\}$.

Keywords: Traditional Eulerian numbers, general Eulerian numbers, permutation, weak excedance.

1 Introduction

Definition 1.1 Given a positive integer n , define Ω_n as the set of all permutations of $[n] = \{1, 2, 3, \dots, n\}$. For a permutation $\pi = p_1 p_2 p_3 \dots p_n \in \Omega_n$, i is called an ascent of π if $p_i < p_{i+1}$; i is called a weak excedance of π if $p_i \geq i$.

It is well known that a traditional Eulerian number $A_{n,k}$ is the number of permutations $\pi \in \Omega_n$ that have k weak excedances ([1], page 215). And $A_{n,k}$ satisfies the recurrence: $A_{n,1} = 1$, ($n \geq 1$), $A_{n,k} = 0$ ($k > n$)

$$A_{n,k} = kA_{n-1,k} + (n+1-k)A_{n-1,k-1} \quad (1 \leq k \leq n) \quad (1)$$

Besides the recursive formula (1), $A_{n,k}$ can be calculated directly by the following analytic formula ([2], page 8):

$$A_{n,k} = \sum_{i=0}^{k-1} (-1)^i (k-i)^n \binom{n+1}{i} \quad (1 \leq k \leq n) \quad (2)$$

Definition 1.2 Given a permutation $\pi = p_1 p_2 p_3 \dots p_n \in \Omega_n$, define functions

$$maj \pi = \sum_{p_j > p_{j+1}} j \quad \text{and}$$

$$a(n, k, i) = \#\{\pi \mid maj \pi = i \text{ \& \& } \pi \text{ has } k \text{ ascents}\}$$

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Since the 1950's, Carlitz ([3],[4]) and his successors have generalized Euler's results to q -sequences $\{1, q, q^2, q^3, \dots\}$. Under Carlitz's definition, the q -Eulerian numbers $A_{n,k}(q)$ are given by

$$A_{n,k}(q) = q^{\frac{(m-k+1)(m-k)}{2}} \sum_{i=0}^{k(n-k-1)} a(n, n-k, i) q^i \quad (3)$$

where functions $a(n, k, i)$ are as defined in Definition 1.2.

In [6], instead of studying q -sequences, the authors have generalized Eulerian numbers to any general arithmetic progression

$$\{a, a + d, a + 2d, a + 3d, \dots\} \quad (4)$$

Under the new definition, and given an arithmetic progression as defined in (4), the general Eulerian numbers $A_{n,k}(a, d)$ can be calculated directly by the following equation ([6], Lemma 2.6)

$$A_{n,k}(a, d) = \sum_{i=0}^k (-1)^i [(k+1-i)d - a]^n \binom{n+1}{i} \quad (5)$$

Interested readers can find more results about the general Eulerian numbers and even general Eulerian polynomials in [6].

2 Combinatorial Interpretation of General Eulerian Numbers

The following concepts and properties will be heavily used in this section.

Definition 2.1 Let $W_{n,k}$ be the set of n -permutations with k weak excedances. Then $|W_{n,k}| = A_{n,k}$. Furthermore, given a permutation $\pi = p_1 p_2 p_3 \dots p_n$, let $Q_n(\pi) = i$ where $p_i = n$.

Given a permutation $\pi \in \Omega_n$, it is known that π can be written as a one line form like $\pi = p_1 p_2 p_3 \dots p_n$. Or π can be written in a disjoint union of distinct cycles. For π written in a cycle form, we can use a *standard representation* by writing (a) each cycle starting with its largest element, and (b) the cycles are written in increasing order of their largest element. Moreover, given a permutation π written in a standard representation cycle form, define a function f as $f(\pi)$ to be the permutation obtained from π by erasing the parentheses. Then f is known as the *fundamental bijection* from Ω_n to itself ([5], page 30). Indeed, the inverse map f^{-1} of the fundamental bijection function f is also famous in illustrating the relation between the ascents and weak excedances as following: ([2], page 98)

Proposition 2.2 The function f^{-1} gives a bijection between the set of permutations on $[n]$ with k ascents and the set $W_{n,k+1}$.

Example 1 The standard representation of permutation $\pi = 5243716$ is $(2)(43)(7615) \in \Omega_7$, and $f(\pi) = 2437615$; $Q_7(\pi) = 5$; $\pi = 5243716$ has 3 ascents, while $f^{-1}(\pi) = (5243)(716) = 6453271 \in W_{7,4}$ has $3+1=4$ weak excedances because $p_1 = 6 > 1$, $p_2 = 4 > 2$, $p_3 = 5 > 3$, and $p_6 = 7 > 6$.

Now suppose we want to construct a sequence consisting of k vertical bars and the first n positive integers. Then the k vertical bars divide these n numbers into $k+1$ compartments. In each compartment, there is either no number or all the numbers are listed in a decreasing order. The following definition is analogous to the definition of [2], page 8.

Definition 2.3 A bar in the above construction is called extraneous if either

- (a) it is immediately followed by another bar; or
- (b) after removing it each of the rest compartment either is empty or consists of integers in a decreasing order.

Example 2 Suppose $n = 7$, $k = 4$, then in the following arrangement

$$32 \mid 1 \mid \mid 7654 \mid$$

the 1st, 2nd, and the 4th bars are extraneous.

Now we are ready to give combinatorial interpretations to the general Eulerian numbers $A_{n,k}(a, d)$. First note that equation (5) implies that $A_{n,k}(a, d)$ is a homogeneous polynomial of degree n with respect to a and d . Indeed,

$$\begin{aligned} A_{n,k}(a, d) &= \sum_{i=0}^k (-1)^i [(k+1-i)d - a]^n \binom{n+1}{i} \\ &= \sum_{i=0}^k (-1)^i [(k+1-i)(d-a) + (k-i)a]^n \binom{n+1}{i} \\ &= \sum_{j=0}^n \left[\sum_{i=0}^k (-1)^i (k+1-i)^{n-j} (k-i)^j \binom{n+1}{i} \right] \binom{n}{j} (d-a)^{n-j} a^j \\ &= \sum_{j=0}^n c_{n,k}(j) \binom{n}{j} (d-a)^{n-j} a^j \end{aligned} \quad (6)$$

where

$$c_{n,k}(j) = \sum_{i=0}^k (-1)^i (k+1-i)^{n-j} (k-i)^j \binom{n+1}{i}, 0 \leq j \leq n. \quad (7)$$

The following Theorem gives combinatorial interpretations to the coefficients $c_{n,k}(j)$, $0 \leq j \leq n$.

Theorem 2.4 Let the general Eulerian numbers $A_{n,k}(a, d)$ be written as in equation (6). Then

$$c_{n,k}(j) = \#\{\pi \in W_{n,k+1} \text{ and } j < Q_n(\pi) \leq n\} + \#\{\pi \in W_{n,k} \text{ and } 1 \leq Q_n(\pi) \leq j\} \quad (8)$$

Proof. We can check the result in (8) for two special values $j = 0$ and $j = n$ quickly. By equation (2),

$$\text{when } j = 0, c_{n,k}(0) = \sum_{i=0}^k (-1)^i (k+1-i)^n \binom{n+1}{i} = A_{n,k+1};$$

$$\text{when } j = n, c_{n,k}(n) = \sum_{i=0}^k (-1)^i (k-i)^n \binom{n+1}{i} = A_{n,k}. \text{ Therefore, (8) is true for } j = 0 \text{ and } j = n.$$

Generally, for $1 \leq j \leq n-1$, we write down k bars with $k+1$ compartments in between. Place each element of $[n]$ in a compartment. If none of the k bars is extraneous, then the arrangement corresponds to a permutation with k ascents. Let B be the set of arrangements with at most one extraneous bar at the end and none of integers $\{1, 2, \dots, j\}$ locating in the last compartment. We will show that $c_{n,k}(j) = |B|$.

To achieve that goal, we use the Principle of Inclusion and Exclusion. There are $(k+1)^{n-j} k^j$ ways to put n numbers into $k+1$ compartments with elements $\{1, 2, \dots, j\}$ avoiding the last compartments. Let B_i be the number of arrangements with (1) none of $\{1, 2, \dots, j\}$ sitting in the last compartment; (2) there are at least i “separating” extraneous bars. Two extraneous bars are separating means that the two bars are not next to each other. Then the Principle of Inclusion and Exclusion shows that

$$|B| = (k+1)^{n-j} k^j - B_1 + B_2 - \dots + (-1)^k B_k \quad (9)$$

Now we consider the value of B_i , where $1 \leq i \leq k$. Suppose that we have $k+1-i$ compartments with $k-i$ bars in between. There are $(k+1-i)^{n-j} (k-i)^j$ ways to insert n numbers into these $k+1-i$ compartments

with first j integers avoiding the last compartment, and list integers in each component in a decreasing order. Then insert i separating extraneous bars into $n + 1$ positions. So we get

$$B_i = (k + 1 - i)^{n-j} (k - i)^j \binom{n+1}{i} \quad (10)$$

Plug the formula (10) into equation (9), we have $c_{n,k}(j) = |B|$.

Given an arrangement $\pi \in B$, if we remove the bars, then we obtain a permutation $\pi \in \Omega_n$. So without confusion, we just use the same notation π to represent both an arrangement in set B and a permutation on $[n]$. Now for each $\pi \in B$, π either

case 1 has no extraneous bar and none of $\{1, 2, \dots, j\}$ locates in the last compartment; Or

case 2 has only one extraneous bar at the end.

If π is in case 1, then π has k ascents since each bar is non-extraneous. And the last compartment of π is nonempty. Therefore the last cycle of $f^{-1}(\pi)$ has to be $(n \dots p_g)$. In other words, $Q_n(f^{-1}(\pi)) = p_g > j$ since none of $\{1, 2, \dots, j\}$ locates in the last compartment. And by Proposition 2.2, $f^{-1}(\pi) \in W_{n,k+1}$.

If π is in case 2, then π has $k - 1$ ascents since only the last bar is extraneous. Note that in this case, the arrangement with no elements of $\{1, 2, \dots, j\}$ in the compartment second to the last, or the last non-empty compartment have been removed by the Principle of Inclusion and Exclusion. Equivalently, at least one number of $\{1, 2, \dots, j\}$ has to be in the compartment second to the last. So the last cycle of $f^{-1}(\pi)$ has to be $(n \dots p_l)$, and $Q_n(f^{-1}(\pi)) = p_l \leq j$. Also by Proposition 2.2, $f^{-1}(\pi) \in W_{n,k}$.

Combing all the results above, statement (8) is correct. \square

The next Theorem describes some interesting properties of the coefficients $c_{n,k}$.

Theorem 2.5 *Let the coefficients $c_{n,k}$ be as described in Theorem 2.4. Then*

1. $\sum_{k=0}^n c_{n,k}(j) = n!$, for any $0 \leq j \leq n$;
2. $c_{n,k}(j) = c_{n,n-k}(n - j)$, for all $0 \leq j, k \leq n$

Before we can prove Theorem 2.5, we need the following Lemma which is also interesting by itself.

Lemma 2.6 *Given a positive integer n , then*

$$\#\{\pi \in W_{n,k} \ \& \ Q_n(\pi) = j\} = \#\{\pi \in W_{n,n+1-k} \ \& \ Q_n(\pi) = n + 1 - j\}$$

for any $1 \leq k, j \leq n$.

Proof. First of all, given a positive integer n , we define a function $g : \Omega_n \rightarrow \Omega_n$ as following:

$$\text{for } \pi = p_1 p_2 \dots p_n \in \Omega_n, \quad g(\pi) = (n + 1 - p_1)(n + 1 - p_2) \dots (n + 1 - p_n)$$

For instance, for $\pi = 53214 \in \Omega_5$, $g(\pi) = 13452$. g is obviously a bijection of Ω_n to itself.

Now for some fixed $1 \leq k, j \leq n$, suppose $S_{k,j} = \{\pi \in W_{n,k} \ \& \ Q_n(\pi) = j\}$, and $T_{k,j} = \{\pi \in W_{n,n+1-k} \ \& \ Q_n(\pi) = n + 1 - j\}$. For any $\pi \in S_{k,j}$ written in the standard representation cycle form. So $\pi = (p_u \dots) \dots (n \dots j)$ and $f(\pi) = p_u \dots n \dots j$ has $k - 1$ ascents by Proposition 2.2. Now we compose $f(\pi)$ with the bijection function g as just defined. Then $g(f(\pi)) = n + 1 - p_u \dots 1 \dots n + 1 - j$ has $n - k$ ascents, which implies that $f^{-1}(g(f(\pi)))$ has $n + 1 - k$ weak excedances. So $f^{-1}(g(f(\pi))) \in W_{n,n+1-k}$. Note that the last cycle of $f^{-1}(g(f(\pi)))$ has to be $(n \dots n + 1 - j)$. Therefore $f^{-1}(g(f(\pi))) \in T_{k,j}$. Since both f and g are bijection functions, $f^{-1}gf$ gives a bijection between $S_{k,j}$ and $T_{k,j}$. \square

Now we are ready to prove Theorem 2.5.

Proof. (of Theorem 2.5) For part 1, by Theorem 2.4,

$$\begin{aligned} \sum_{k=0}^n c_{n,k}(j) &= \sum_{k=0}^n \#\{\pi \in W_{n,k+1} \text{ and } j < Q_n(\pi) \leq n\} + \sum_{k=0}^n \#\{\pi \in W_{n,k} \text{ and } 1 \leq Q_n(\pi) \leq j\} \\ &= \sum_{k=0}^n \#\{\pi \in W_{n,k}\} = |\Omega_n| = n! \end{aligned}$$

For part 2, also by Theorem 2.4,

$$\begin{aligned} c_{n,k}(j) &= \sum_{i=j+1}^n \#\{\pi \in W_{n,k+1} \text{ and } Q_n(\pi) = i\} + \sum_{m=1}^j \#\{\pi \in W_{n,k} \text{ and } Q_n(\pi) = m\} \\ &= \sum_{i=j+1}^n \#\{\pi \in W_{n,n-k} \text{ and } Q_n(\pi) = n+1-i\} \\ &\quad + \sum_{m=1}^j \#\{\pi \in W_{n,n+1-k} \text{ and } Q_n(\pi) = n+1-m\} \quad \text{by Lemma 2.6} \\ &= \#\{\pi \in W_{n,k} \text{ and } 1 \leq Q_n(\pi) \leq n-j\} + \#\{\pi \in W_{n,n+1-k} \text{ and } n-j < Q_n(\pi) \leq n\} \\ &= c_{n,n-k}(n-j) \end{aligned}$$

□

Remark Using the analytic formula of $c_{n,k}(j)$ as in (7), part 2 of Theorem 2.5 implies the following identity:

$$\sum_{i=0}^k (-1)^i (k+1-i)^{n-j} (k-i)^j \binom{n+1}{i} = \sum_{l=0}^{n-k} (-1)^l (n+1-k-l)^j (n-k-l)^{n-j} \binom{n+1}{l},$$

where n is a positive integer, and $0 \leq j, k \leq n$.

3 Another Combinatorial Interpretation of $c_{n,k}(1)$ and $c_{n,k}(n-1)$

In pursuing the combinatorial meanings of the coefficients $c_{n,k}$, the authors have found some other interesting properties about permutations. The results in this section will reveal close connections between the traditional Eulerian numbers $A_{n,k}$ and $c_{n,k}(j)$, where $j = 1$ or $j = n-1$.

One fundamental concept of permutation combinatorics is *inversion*. A pair (p_i, p_j) is called an *inversion* of the permutation $\pi = p_1 p_2 \dots p_n$ if $i < j$ and $p_i > p_j$ ([5], page 36). The following definition provides the main concepts of this section.

Definition 3.1 For a fixed positive integer n , let $AW_{n,k} = \{\pi = p_1 p_2 p_3 \dots p_n \mid \pi \in W_{n,k} \text{ and } p_1 < p_n\}$ (or (p_1, p_n) is not an inversion), and $BW_{n,k} = W_{n,k} \setminus AW_{n,k}$ (or (p_1, p_n) is an inversion).

It is obvious that $|AW_{n,k}| + |BW_{n,k}| = A_{n,k}$. The following Theorem interprets coefficients $c_{n,k}(1)$ and $c_{n,k}(n-1)$ in terms of $AW_{n,k}$ and $BW_{n,k}$.

Theorem 3.2 Let the coefficients $c_{n,k}$ of the general Eulerian numbers be written as in equation (7). $AW_{n,k}$ and $BW_{n,k}$ are as defined in Definition 3.1. Then

$$(1) \quad c_{n,k}(1) = 2|AW_{n,k+1}|;$$

$$(2) \quad c_{n,k}(n-1) = 2|BW_{n,k}|.$$

Proof. For part (1). By Theorem 2.4, $c_{n,k}(1) = |S_1| + |S_2|$, where $S_1 = \{\pi = p_1 p_2 \dots p_n \mid \pi \in W_{n,k+1} \text{ and } p_1 \neq n\}$, $S_2 = \{\pi = p_1 p_2 \dots p_n \mid \pi \in W_{n,k} \text{ and } p_1 = n\}$. Given a permutation $\pi = p_1 p_2 \dots p_n \in S_1$ and $p_n \neq n$, then both $p_1 p_2 \dots p_n$ and $p_n p_2 \dots p_1$ belong to S_1 , so one of them has to be in $AW_{n,k+1}$; If $\pi = p_1 p_2 \dots p_n \in S_1$ and $p_n = n$, then $\pi \in AW_{n,k+1}$, but $p_n p_2 \dots p_1 \in S_2$. Therefore, $\frac{1}{2}c_{n,k}(1) = |AW_{n,k+1}|$.

Part (2) can be proved using exactly the same method. So we leave it to the readers as an exercise. \square

$|AW_{n,k}|$ and $|BW_{n,k}|$ are interesting combinatorial concepts by themselves. Note that generally speaking, $|AW_{n,k}| \neq |BW_{n,k}|$. Indeed, $|AW_{n,k}| = |BW_{n,n+1-k}|$.

Theorem 3.3 *For any positive integer $n \geq 2$, the sets $AW_{n,k}$ and $BW_{n,k}$ are defined in Definition 3.1. Then $|AW_{n,k}| = |BW_{n,n+1-k}|$ for $1 \leq k \leq n$.*

Proof. It is an obvious result of part 2 of Theorem 2.5 and Theorem 3.2. \square

Our last result of this paper is the following Theorem which reveals that both $|AW_{n,k}|$ and $|BW_{n,k}|$ take exactly the same recursive formula as the traditional Eulerian numbers $A_{n,k}$ as shown in equation (1).

Theorem 3.4 *For a fixed positive integer n , let $AW_{n,k}$ and $BW_{n,k}$ be as defined in Definition 3.1, then*

$$k|AW_{n-1,k}| + (n+1-k)|AW_{n-1,k-1}| = |AW_{n,k}| \quad \text{and} \quad (11)$$

$$k|BW_{n-1,k}| + (n+1-k)|BW_{n-1,k-1}| = |BW_{n,k}| \quad (12)$$

Proof. A computational proof can be obtained straightforward by using equation (7) and Theorem 3.2. But here we provide a proof in a flavor of combinatorics.

Idea of the proof: For equation (11), given a permutation $A_1 = p_1 p_2 p_3 \dots p_{n-1} \in AW_{n-1,k}$, for each position i with $p_i \geq i$, we insert n into a certain place of A_1 , such that the new permutation A'_1 is in $AW_{n,k}$. There are k such positions, so we can get k new permutations in $AW_{n,k}$. Similarly, if $A_2 = p_1 p_2 p_3 \dots p_{n-1} \in AW_{n-1,k-1}$, for each position i with $p_i < i$, and the position at the end of A_2 , we insert n into a specific position of A_2 and the resulting new permutation A'_2 is in $AW_{n,k}$. There are $n+1-k$ such positions, so we can get $n+1-k$ new permutations in $AW_{n,k}$. We will show that all the permutations obtained from the above constructions are distinct, and they have exhausted all the permutations in $AW_{n,k}$.

For any fixed $A' = \pi_1 \pi_2 \pi_3 \dots \pi_n \in AW_{n,k}$, then $\pi_1 < \pi_n$. We classify A' into the following disjoint cases:

case a. $\pi_i = n$ with $i < n$. So $A' = \pi_1 \pi_2 \dots \pi_{i-1} n \pi_{i+1} \dots \pi_{n-1} \pi_n$.

- a.1 $\pi_1 < \pi_{n-1}$, and $\pi_n \geq i$;
- a.2 $\pi_1 < \pi_{n-1}$, and $\pi_n < i$;
- a.3 $\pi_1 > \pi_{n-1}$, $\pi_n < n-1$, and $\pi_n \geq i$;
- a.4 $\pi_1 > \pi_{n-1}$, $\pi_n < n-1$, and $\pi_n < i$;
- a.5 $\pi_1 > \pi_{n-1}$, and $\pi_n = n-1$;

case b. $\pi_n = n$. So $\pi_i = n-1$ for some $i < n$ and $A' = \pi_1 \pi_2 \dots \pi_{i-1} n-1 \dots \pi_{n-1} n$.

- b.1 $\pi_1 < \pi_{n-1}$;
- b.2 $\pi_{n-1} < \pi_1 < n-1$, and $\pi_{n-1} \geq i$;
- b.3 $\pi_{n-1} < \pi_1 < n-1$, and $\pi_{n-1} < i$;
- b.4 $\pi_1 = n-1$.

Based on the classifications listed above, we can construct a map $f : \{AW_{n-1,k}, AW_{n-1,k-1}\} \rightarrow AW_{n,k}$ by applying the idea of the proof we have illustrated at the beginning of the proof. To save space, the map f is demonstrated in Table 1. From Table 1 we can see that in each case, the positions of inserting n are all different. So all the images obtained in a certain case are different. Since all the cases are disjoint, all the images $A' \in AW_{n,k}$ are distinct.

Similarly, for each $B' = \pi_1 \pi_2 \pi_3 \dots \pi_n \in BW_{n,k}$, then $\pi_1 > \pi_n$. We classify B' into the following disjoint cases:

case c. $\pi_i = n$ with $1 < i \leq n-1$. So $B' = \pi_1 \pi_2 \dots \pi_{i-1} n \pi_{i+1} \dots \pi_{n-1} \pi_n$.

$A = p_1 p_2 \dots p_{n-1}$	Position i	Condition	$A' \in AW_{n,k}$
$A \in AW_{n-1,k}$	$1 < i \leq n-1$ and $p_i \geq i$	$p_i > p_1$	$A' = p_1 p_2 \dots p_{i-1} n p_{i+1} \dots p_{n-1} p_i$, with $p_1 < p_{n-1}$ and $p_1 < p_i$. (Case a.1)
		$p_i < p_1$ and	$A' = p_1 p_2 \dots p_{i-1} n p_{i+1} \dots p_{n-2} p_i p_{n-1}$, with $p_i < p_1 < p_{n-1} < n-1$. (Case a.3)
		$p_i < p_1$ and	$A' = p_1 p_2 \dots p_{i-1} n-1 p_{i+1} \dots p_{n-2} p_i n$, with $p_i < p_1$ and $p_i \geq i$. (Case b.2)
	$i = 1$	$p_i = n-1$ and $i < n-1$	$A' = p_{n-1} p_2 \dots p_{i-1} n p_{i+1} \dots p_{n-2} p_1 n-1$, with $p_1 < p_{n-1}$. (Case a.5)
		$p_{n-1} = n-1$	$A' = n-1 p_2 \dots p_{n-2} p_1 n$, (Case b.4)
$A \in AW_{n-1,k-1}$	$1 < i \leq n-1$ and $p_i < i$	$p_i > p_1$	$A' = p_1 p_2 \dots p_{i-1} n p_{i+1} \dots p_{n-1} p_i$, with $p_1 < p_{n-1}$ and $p_1 < p_i$. (Case a.2)
		$p_i < p_1$ and $p_{n-1} < n-1$	$A' = p_1 p_2 \dots p_{i-1} n p_{i+1} \dots p_{n-2} p_i p_{n-1}$, with $p_i < p_1 < p_{n-1} < n-1$. (Case a.4)
		$p_i < p_1$ and $p_{n-1} = n-1$	$A' = p_1 p_2 \dots p_{i-1} n-1 p_{i+1} \dots p_{n-2} p_i n$. with $p_i < p_1$ and $p_i < i$. (Case b.3)
	$i = n$		$A' = p_1 p_2 \dots p_{n-1} n$. (Case b.1)

Table 1: The Map $f : \{AW_{n-1,k}, AW_{n-1,k-1}\} \rightarrow AW_{n,k}$

- c.1 $\pi_1 > \pi_{n-1}$, and $\pi_n \geq i$;
- c.2 $\pi_1 > \pi_{n-1}$, and $\pi_n < i$;
- c.3 $\pi_1 < \pi_{n-1} < n-1$, $\pi_{n-1} \geq i$;
- c.4 $\pi_1 < \pi_{n-1} < n-1$, $\pi_{n-1} < i$;
- c.5 $\pi_{n-1} = n-1$;
- c.6 $\pi_{n-1} = n$;

case d. $\pi_1 = n$. So $B' = n \pi_2 \dots \pi_{n-2} \pi_{n-1}$.

- d.1 $\pi_{n-2} < \pi_{n-1}$;
- d.2 $\pi_{n-2} > \pi_{n-1}$.

To prove equation (12), we use a similar idea of proof as shown above. If $B_1 = p_1 p_2 p_3 \dots p_{n-1} \in BW_{n-1,k}$, for each position i with $p_i \geq i$, we insert n into a certain place of B_1 to get $B'_1 \in AW_{n,k}$; If $B_2 = p_1 p_2 p_3 \dots p_{n-1} \in$

$BW_{n-1,k-1}$, for each position i with $p_i < i$, and the position i where $p_i = n - 1$, we insert n into a specific position of B_2 to obtain $B'_2 \in AW_{n,k}$. Such a map $g : \{BW_{n-1,k}, BW_{n-1,k-1}\} \rightarrow BW_{n,k}$ is illustrated in Table 2. And the distinct images under g exhaust all the permutations in $BW_{n,k}$.

$B = p_1 p_2 \dots p_{n-1}$	Position i	Condition	$B' \in BW_{n,k}$
$B \in BW_{n-1,k}$	$1 < i < n - 1$ and $p_i \geq i$	$p_1 > p_i$	$B' = p_1 \dots p_{i-1} n p_{i+1} \dots p_{n-1} p_i$, with $p_1 > p_{n-1}$ and $p_i \geq i$. (Case c.1)
		$p_1 < p_i < n - 1$	$B' = p_1 \dots p_{i-1} n p_{i+1} \dots p_i p_{n-1}$, with $p_1 < p_i < n - 1$ and $p_i \geq i$. (Case c.3)
		$p_1 < p_i = n - 1$	$B' = n p_2 \dots p_{i-1} n - 1 p_{i+1} \dots p_1 p_{n-1}$, with $p_i = n - 1$ and $p_1 > p_{n-1}$. (Case d.2)
	$i = 1$	$p_1 \geq 1$	$B' = n p_2 \dots p_{n-1} p_1$, with $p_{n-1} < p_1$. (Case d.1)
$B \in BW_{n-1,k-1}$	$1 < i < n - 1$ and $p_i < i$	$p_1 > p_i$	$B' = p_1 \dots p_{i-1} n p_{i+1} \dots p_{n-1} p_i$, with $p_1 > p_{n-1}$ and $p_i < i$. (Case c.2)
		$p_1 < p_i$	$B' = p_1 \dots p_{i-1} n p_{i+1} \dots p_i p_{n-1}$, with $p_1 < p_i < n - 1$ and $p_i < i$. (Case c.4)
	$i = n - 1$ $p_i < i$	$p_1 > p_i = p_{n-1}$	$B' = p_1 p_2 \dots p_{n-2} n p_{n-1}$. (Case c.6)
	$1 \leq i < n - 1$ and $p_i = n - 1$	$p_i = n - 1$	$B' = p_1 p_2 \dots p_{i-1} n p_{i+1} \dots p_{n-2} n - 1 p_{n-1}$. (Case c.5)

Table 2: The Map $g : \{BW_{n-1,k}, BW_{n-1,k-1}\} \rightarrow BW_{n,k}$

Here is a concrete example for the constructions illustrated in Table 2:

Example 3 Suppose $n = 4$, $k = 2$. We want to obtain $BW_{4,2} = \{3142, 3412, 3421, 4132, 4213, 4312, 4321\}$ from $BW_{3,2} = \{321, 231\}$ and $BW_{3,1} = \{312\}$. For $321 \in BW_{3,2}$, $p_1 = 3 \geq 1$, then it corresponds to $B' = 4213$ which is Case d.1 in Table 2; $p_2 = 2 \geq 2$, then it corresponds to $B' = 3412$ which is Case c.1 in Table 2. Similarly, we can construct $\{4312, 4321\}$ from $231 \in BW_{3,2}$, and $\{3421, 3142, 4132\}$ from $312 \in BW_{3,1}$ using Table 2.

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